Analyticity of Power Series

Termwise Differentiation and Taylor Coefficients

Lecture Notes

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1 Analyticity of Power Series

Lemma

Let $|z| \le r$, $|w| \le r$. Then $|z^n - w^n| \le n|z-w|r^{n-1}$.

Proof. $z^n - w^n = (z-w)(z^{n-1} + z^{n-2}w + \cdots + w^{n-1})$, so

$$|z^n - w^n| \le |z-w| \sum_{k=0}^{n-1} \underbrace{|w^k z^{n-1-k}|}_{\le r^{n-1}} \le n\, r^{n-1}|z-w|. \qquad \square$$

Theorem (analyticity of power series)

Let $f(z) = \sum_{n=0}^{\infty} a_n z^n$ be a power series with radius of convergence R. Then f is analytic on $B(0,R)$ and

$$f'(z) = \sum_{n=1}^{\infty} n a_n z^{n-1} \qquad \text{(termwise differentiation)}.$$

Proof. Let $f_1(z) := \sum_{n=1}^{\infty} n a_n z^{n-1}$.

Step 1: f_1 has the same radius of convergence R. Let R_1 be the radius of convergence of $\sum_{n=1}^{\infty} n a_n z^{n-1}$. Since $|na_n| \ge |a_n|$, we get $R_1 \le R$. Conversely, if $r < R$, pick r_0 with $r < r_0 < R$; then

$$\lim_{n\to\infty} n|a_n| r^n = \lim_{n\to\infty} \left(|a_n| r_0^n\right) \cdot \left(n\left(\tfrac{r}{r_0}\right)^n\right) = 0,$$

since $|a_n| r_0^n$ is bounded (as $r_0 < R$) and $nq^n \to 0$ for $q = r/r_0 < 1$ (shown below). So $(n|a_n|r^n)$ is bounded, hence $r \le R_1$. As this holds for all $r < R$, $R \le R_1$, so $R_1 = R$.

Aside: $nq^n \to 0$ for $0 \le q < 1$. Write $q = 1/(1+\delta)$, $\delta > 0$. By the binomial theorem, $(1+\delta)^n > \binom{n}{2}\delta^2 = \frac{n(n-1)}{2}\delta^2$, so

$$nq^n < \frac{2n}{n(n-1)\delta^2} = \frac{2}{\delta^2(n-1)} \to 0.$$

Step 2: the difference quotient. Write $f(z) = S_n(z) + R_n(z)$, where $S_n(z) = \sum_{k=0}^{n} a_k z^k$ and $R_n(z) = \sum_{k=n+1}^{\infty} a_k z^k$; let $S_n'(z) = \sum_{k=1}^{n} k a_k z^{k-1}$ be the n-th partial sum of f_1. Fix $|z|, |z_0| < r < R$. Then

$$\frac{f(z) - f(z_0)}{z - z_0} - f_1(z_0) = \underbrace{\left(\frac{S_n(z) - S_n(z_0)}{z - z_0} - S_n'(z_0)\right)}_{\text{I}} + \underbrace{\left(S_n'(z_0) - f_1(z_0)\right)}_{\text{II}} + \underbrace{\frac{R_n(z) - R_n(z_0)}{z - z_0}}_{\text{III}}.$$

By the Lemma,

$$|\mathrm{III}| = \left| \sum_{k=n+1}^{\infty} a_k \frac{z^k - z_0^k}{z - z_0} \right| \leq \sum_{k=n+1}^{\infty} |a_k|\, k\, r^{k-1}.$$

Step 3: assembling the estimate. Fix $\varepsilon > 0$.

- Choose n large enough that $|S_n'(z_0) - f_1(z_0)| < \varepsilon/3$ (partial sums of f_1 converge, since f_1 has radius of convergence R), and $\sum_{k=n+1}^{\infty} k|a_k|r^{k-1} < \varepsilon/3$ (tail of a convergent series, as $r < R = R_1$).
- Then choose $\delta > 0$ such that $|z - z_0| < \delta \implies |\mathrm{I}| < \varepsilon/3$, using differentiability of the polynomial S_n.

For $|z - z_0| < \delta$,

$$\left| \frac{f(z) - f(z_0)}{z - z_0} - f_1(z_0) \right| \leq |\mathrm{I}| + |\mathrm{II}| + |\mathrm{III}| < \varepsilon.$$

So $\lim_{z \to z_0} \dfrac{f(z) - f(z_0)}{z - z_0} = f_1(z_0)$, i.e. $f'(z_0) = f_1(z_0)$ for every $z_0 \in B(0, R)$. □

Corollary (Taylor coefficients)

$f(z) = \sum a_n z^n$ is infinitely differentiable on $B(0, R)$, with

$$f^{(k)}(z) = \sum_{n=k}^{\infty} \frac{n!}{(n-k)!}\, a_n z^{n-k}, \qquad \text{and in particular} \qquad a_n = \frac{f^{(n)}(0)}{n!}.$$

Proof. Induction on k, applying the Theorem repeatedly to the (termwise-differentiated) power series, and setting $z = 0$ to read off $f^{(k)}(0) = k!\, a_k$. □